# A short and elementary proof of the main Bahadur-Kiefer theorem

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# A SHORT AND ELEMENTARY PROOF OF THE MAIN BAHADUR–KIEFER THEOREM

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A short proof of the lower bound in the strong version of the famous Theorem 1A in Kiefer (1970) on the Bahadur–Kiefer process is presented. The proof is elementary and, in particular, does not use strong approximations.

Let $U_1, U_2, \dots$ be a sequence of independent uniform-(0,1) random variables and, for each $n \in \mathbb{N}$, let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[0,t]}(U_i), \qquad 0 \le t \le 1,$$

be the empirical distribution function at stage $n$. The uniform empirical process will be written as

$$\alpha_n(t) = n^{1/2}(F_n(t) - t), \qquad 0 \le t \le 1; \qquad \alpha_n(t) = 0 \quad \text{for } t < 0 \text{ or } t > 1.$$

Also, for each $n \in \mathbb{N}$,

$$Q_n(t) = \inf\{s\colon F_n(s) \ge t\}, \qquad 0 < t \le 1, \qquad Q_n(0) = 0,$$

denotes the empirical quantile function, and we write

$$\beta_n(t) = n^{1/2}(Q_n(t) - t), \qquad 0 \le t \le 1,$$

for the corresponding uniform quantile process. The so-called Bahadur–Kiefer process is defined by

$$R_n(t) = \alpha_n(t) + \beta_n(t), \qquad 0 \le t \le 1.$$

This process is introduced in Bahadur (1966); in Kiefer [(1970), Theorem 1A] the "in-probability-analogue" of the following statement is proved:

$$\lim_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}} \frac{\|R_n\|}{\|\alpha_n\|^{1/2}} = 1 \quad \text{a.s.,} \tag{1}$$

where $\|f\| = \sup_{0 \le t \le 1} |f(t)|$ for any real-valued function $f$ on [0,1]. In the latter paper a proof of (1) itself is claimed but not presented. However, it is proved in Shorack (1982) that, indeed, the expression on the left in (1) (with lim replaced by lim sup) is not larger than 1, almost surely (note that $\|\alpha_n\| = \|\beta_n\|$), whereas in a recent paper by Deheuvels and Mason (1990) it is established that the same expression is not smaller than 1, almost surely.

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The short and elegant proof in Shorack (1982) is based on the Kiefer process strong approximation of $\alpha_n$, but in Shorack and Wellner [(1986), pages 590–591] a similar, direct proof of the "upper-bound part" is given. The ingenious and generally applicable proof of the "lower-bound part" [which finally led to a complete proof of (1)] in Deheuvels and Mason (1990) is very technical; moreover, it is again based on a strong approximation of $\alpha_n$.

It is the purpose of this note to give a new, short proof of the lower-bound part of (1). That is, we will prove that

$$\liminf_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}} \frac{\|R_n\|}{\|\alpha_n\|^{1/2}} \geq 1 \quad \text{a.s.} \tag{2}$$

Our proof is rather easy and not based on strong approximations. It uses as tools the following well-known facts on empirical and quantile processes, although most of them are not required at their full strength.

FACT 1 [Mogul'skii (1979)]. We have

$$\liminf_{n\to\infty} (\log\log n)^{1/2} \|\alpha_n\| = \frac{\pi}{8^{1/2}} \quad \text{a.s.} \tag{3}$$

FACT 2 (Easy). We have

$$\|\beta_n + \alpha_n \circ Q_n\| = n^{-1/2} \quad \text{a.s.} \tag{4}$$

FACT 3 [Kiefer (1970)]. We have

$$\limsup_{n\to\infty} n^{1/4} (\log n)^{-1/2} (\log\log n)^{-1/4} \|R_n\| = 2^{-1/4} \quad \text{a.s.} \tag{5}$$

Define the oscillation modulus of $\alpha_n$ by

$$\omega_n(a) = \sup_{\substack{t-s\leq a\\ 0\leq s\leq t\leq 1}} |\alpha_n(t) - \alpha_n(s)|, \qquad 0 < a \leq 1;$$

let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $a_n \downarrow 0$ and $na_n \uparrow$.

FACT 4 [Mason, Shorack and Wellner (1983)]. If $\log(1/a_n)/\log\log n \to c \in [0,\infty)$, then

$$\limsup_{n\to\infty} \frac{\omega_n(a_n)}{(a_n \log\log n)^{1/2}} = (2(1+c))^{1/2} \quad \text{a.s.} \tag{6}$$

FACT 5 [Stute (1982)]. If $\log(1/a_n)/\log\log n \to \infty$ and $na_n/\log n \to \infty$, then

$$\lim_{n\to\infty} \frac{\omega_n(a_n)}{(a_n \log(1/a_n))^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{7}$$

FACT 6 [Mallows (1968)]. If $(N_1, \dots, N_k)$, $k \in \mathbb{N}$, has a multinomial distribution with parameters $m$ and $p_1, \dots, p_k$, where $m \in \mathbb{N}$ and $p_1, \dots, p_k$ are nonnegative with $\sum_{i=1}^k p_i = 1$, then, for all $\lambda_1, \dots, \lambda_k$,

$$P(N_1 \le \lambda_1, \dots, N_k \le \lambda_k) \le \prod_{i=1}^{k} P(N_i \le \lambda_i).$$

FACT 7 [Kolmogorov (1929)]. Let $m \in \mathbb{N}$ and $t \in (0, \frac{1}{2})$. Then for every $\delta > 0$ there exist $K_1, K_2 \in (0, \infty)$ such that, for $K_1 t^{1/2} \le \lambda \le K_2 m^{1/2} t$,

$$P(\alpha_m(t) > \lambda) \ge \exp\left(-\frac{(1+\delta)\lambda^2}{2t(1-t)}\right).$$

FACT 8 [Dvoretzky, Kiefer and Wolfowitz (1956) and Massart (1990)]. Let $n \in \mathbb{N}$. Then, for all $\lambda \ge 0$,

$$P(\|\alpha_n\| \ge \lambda) \le 2\exp(-2\lambda^2). \tag{8}$$

PROOF OF (2). Let $I$ denote the identity function on $[0,1]$. First we will show that, as $n \to \infty$,

$$\left\| \beta_n + \alpha_n \circ \left(I - \frac{\alpha_n}{n^{1/2}}\right) \right\| = \mathscr{O}\left((\log n)^{3/4} (\log\log n)^{1/8} n^{-3/8}\right) \quad \text{a.s.} \tag{9}$$

To prove (9), first observe that by (4) and $Q_n = I + \beta_n / n^{1/2}$ we have that

$$\left\| \beta_n + \alpha_n \circ \left(I + \frac{\beta_n}{n^{1/2}}\right) \right\| = n^{-1/2} \quad \text{a.s.}$$

Now using (5) and (7) yields (9). Observe that it immediately follows from (9) and (3) that for a proof of (2) it is sufficient to show that

$$\liminf_{n \to \infty} \frac{n^{1/4}}{(\log n)^{1/2}} \frac{\|\alpha_n \circ (I - \alpha_n / n^{1/2}) - \alpha_n\|}{\|\alpha_n\|^{1/2}} \ge 1 \quad \text{a.s.} \tag{10}$$

Set, for $0 \le t \le 1$,

$$\bar{\alpha}_n(t) = \begin{cases} \alpha_n(t), & \text{if } |\alpha_n(t)| > \dfrac{1}{\log n}, \\ \dfrac{1}{\log n}, & \text{if } |\alpha_n(t)| \le \dfrac{1}{\log n}. \end{cases}$$

Define the following grid on $[0,1]$: $t_{i,n} = i/[\log n]$, $i = 0, 1, \dots, [\log n]$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. From (3) and (6) we have that

$$\lim_{n \to \infty} \frac{\max_{0 \le i \le [\log n]} |\bar{\alpha}_n(t_{i,n})|}{\|\alpha_n\|} = 1 \quad \text{a.s.}$$

Moreover, from (6), (7) and (3), it follows that

$$\lim_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}}\left[\max_{0\le i\le[\log n-1]}\ \sup_{t_{i,n}\le t\le t_{i,n+1}}\left|\alpha_n\left(t-\frac{\overline{\alpha}_n(t_{i,n})}{n^{1/2}}\right)\right.\right.$$
$$\left.\left.-\alpha_n\left(t-\frac{\alpha_n(t)}{n^{1/2}}\right)\right|\right]\|\alpha_n\|^{-1/2}= 0 \text{ a.s.}$$

Hence, instead of proving (10), it suffices to prove that

$$\liminf_{n\to\infty}\frac{n^{1/4}}{(\log n)^{1/2}}$$
$$\times\frac{\max_{0\le i\le[\log n]-1}\sup_{t_{i,n}\le t\le t_{i+1,n}}\left|\alpha_n\left(t-\overline{\alpha}_n(t_{i,n})/n^{1/2}\right)-\alpha_n(t)\right|}{\max_{0\le i\le[\log n]}|\overline{\alpha}_n(t_{i,n})|^{1/2}} \tag{11}$$
$$\ge 1 \quad \text{a.s.}$$

Using the Borel–Cantelli lemma, a proof of (11) is established if we show that, for all $\varepsilon\in(0,1)$, $\sum_{n=3}^{\infty}PA_n<\infty$, where

$$A_n=\left\{n^{1/4}\max_{0\le i\le[\log n]-1}\ \sup_{t_{i,n}\le t\le t_{i+1,n}}\left|\alpha_n\left(t-\frac{\overline{\alpha}_n(t_{i,n})}{n^{1/2}}\right)-\alpha_n(t)\right|\right.$$
$$\left.\le\left((1-\varepsilon)\max_{0\le i\le[\log n]}|\overline{\alpha}_n(t_{i,n})|\log n\right)^{1/2}\right\}.$$

Write

$$C_n=C_n(c_{1,n},c_{2,n},\dots,c_{[\log n]-1,n})=\{\alpha_n(t_{i,n})=c_{i,n},\ 1\le i\le[\log n]-1\},$$

$c_{i,n}\in[-\log n,\log n]$ and $c_{i,n}$ is such that $nt_{i,n}+n^{1/2}c_{i,n}\in\{0,1,\dots,n\}$ and such that $nt_{i,n}+n^{1/2}c_{i,n}$ is nondecreasing in $i$ (observe that $PC_n>0$). Set

$$\bar{c}_n=\left(\max_{1\le i\le[\log n]-1}|c_{i,n}|\right)\vee\left(\frac{1}{\log n}\right),$$

and, on $C_n$, let $t_n$ be the smallest $t_{i,n}, 0\le i\le[\log n]$, such that $|\overline{\alpha}_n(t_{i,n})|=\bar{c}_n$; write $d_n=\alpha_n(t_n)$ and $\bar{d}_n=\overline{\alpha}_n(t_n)$; set $t_n'=t_n+1/[\log n]$ and $d_n'=\alpha_n(t_n')$. Now we have

$$P(A_n|C_n)\le P\left(n^{1/4}\sup_{\substack{v-u=\bar{c}_n/n^{1/2}\\ t_n\le u\le v\le t_n'}}|\alpha_n(v)-\alpha_n(u)|\right.$$
$$\left.\le((1-\varepsilon)\bar{c}_n\log n)^{1/2}\middle|C_n\right). \tag{12}$$

Write $m_n=n/[\log n]+n^{1/2}(d_n'-d_n)$ and note that, on $C_n$, $m_n=n(F_n(t_n')-F_n(t_n))$; obviously $|m_n[\log n]/n-1|\le 2(\log n)^2n^{-1/2}\to 0$ as $n\to\infty$. Now it is not hard to see that, on $C_n$, the process $\tilde{\alpha}_{m_n}$ defined by

$$\tilde{\alpha}_{m_n}(s) = \left(\frac{n}{m_n}\right)^{1/2}\left\{\alpha_n\left(t_n + \frac{s}{[\log n]}\right) - (d_n(1-s) + d'_n s)\right\}, \qquad 0 \le s \le 1,$$

is a uniform empirical process based on $m_n$ observations. Hence the right-hand side of (12) can be written as

$$P\left(n^{1/4} \sup_{\substack{s-r=\bar{c}_n[\log n]n^{-1/2}\\ 0\le r\le s\le 1}} \left|\left(\frac{m_n}{n}\right)^{1/2}\{\tilde{\alpha}_{m_n}(s) - \tilde{\alpha}_{m_n}(r)\} + \bar{d}_n[\log n](d_n - d'_n)n^{-1/2}\right| \le ((1-\varepsilon)\bar{c}_n \log n)^{1/2}\right). \tag{13}$$

Now observe that

$$\frac{|n^{1/4}\bar{d}_n[\log n](d_n - d'_n)n^{-1/2}|}{(\bar{c}_n \log n)^{1/2}} \le 2\bar{c}_n^{1/2}(\log n)^{3/2}n^{-1/4} \le 2(\log n)^2 n^{-1/4} \to 0 \quad \text{as } n \to \infty.$$

Therefore, for large $n$, the expression in (13) is bounded from above by

$$P\left(n^{1/4}\left(\frac{m_n}{n}\right)^{1/2} \sup_{\substack{s-r=\bar{c}_n[\log n]n^{-1/2}\\ 0\le r\le s\le 1}} |\tilde{\alpha}_{m_n}(s) - \tilde{\alpha}_{m_n}(r)| \le \left(\left(1-\frac{1}{2}\varepsilon\right)\bar{c}_n \log n\right)^{1/2}\right), \tag{14}$$

which by Fact 6 is less than or equal to

$$\left\{P\left(n^{1/4}\left(\frac{m_n}{n}\right)^{1/2}\tilde{\alpha}_{m_n}\left(\frac{\bar{c}_n[\log n]}{n^{1/2}}\right) \le \left(\left(1-\frac{1}{2}\varepsilon\right)\bar{c}_n \log n\right)^{1/2}\right)\right\}^{n^{1/2}/(\log n)^2}. \tag{15}$$

It is easy to check that, for large $n$, Fact 7 applies to the probability in (15). This yields, with $\delta = \varepsilon/4$, the following upper bound for the expression in (15):

$$(1 - n^{-(1-\varepsilon/4)/2})^{n^{1/2}/(\log n)^2} \le \exp\left(\frac{-n^{\varepsilon/8}}{(\log n)^2}\right) \le \frac{1}{n^2}. \tag{16}$$

We are now ready to complete the proof. Combining (12)–(16), we have $P(A_n|C_n) \le 1/n^2$ ($n$ large). Set $D_n = \{\|\alpha_n\| > \log n\}$ and note that (8) implies that $PD_n \le 1/n^2$ ($n \ge 4$). Hence, for large $n$,

$$\begin{aligned} PA_n &\le P(A_n \cap D_n^c) + PD_n \le \left(\sup{}^* P(A_n|C_n)\right) + PD_n \\ &\le \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}, \end{aligned} \tag{17}$$

where $\sup^*$ denotes the supremum over all $C_n$ as defined before. Now, of course, $\sum_{n=3}^{\infty} PA_n < \infty$ because of (17). This proves (11) and hence (2). □

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